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Stochastic Quantization of Time-Dependent Systems by the Haba and Kleinert Method

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The stochastic quantization method recently developed by Haba and Kleinert is extended to non-autonomous mechanical systems, in the case of the time-dependent harmonic oscillator. In comparison with the autonomous case, the quantization procedure involves the solution of a nonlinear, auxiliary equation. Using a rescaling transformation, the Schrödinger equation for the time-dependent harmonic oscillator is obtained after averaging of a classical stochastic differential equation.

KEY WORDS: stochastic quantization; time-dependent harmonic oscillator; Schrödinger equation; classical simulation of quantum systems.
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1. INTRODUCTION

The fundamental goal of stochastic quantization (Bacciagaluppi, 1999; Belavkin, 2003; de la Peña and Cetto, 1996; Gaioli *et al*., 1997; Hooft, 1997; Namiki, 1982, 2000; Olavo, 2000; Torres and Figueiredo, 2003) is to reproduce the whole content of quantum mechanics by means of classical equations subjected to random perturbations. Presently, there remains the controversy about the physical origin of the noise, which is supposed to arise, for instance, from a fluctuating metrics or from fluctuations of the vacuum electromagnetic field. In spite of this, stochastic quantization is an attractive alternative for all physicists felling not so comfortable with the epistemological content of the traditional interpretation of quantum mechanics.

Recently, Haba, and Kleinert (2002) have proposed a new approach for stochastic quantization, hereafter referred to as the (Haba and Kleinert) HK method. HK is based directly on the use of Newton equations with the presence of noise. After introducing an auxiliary field, defined in terms of the solutions for the stochastic dynamical equations, a deterministic partial differential equation is found averaging over all stochastic processes. This deterministic equation, after

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a suitable transformation, yields the Schödinger equation for the mechanical system under consideration. Solving the Schrödinger equation reproduces the spectrum of the system. In comparison with other stochastic quantization methods like Nelson's stochastic mechanics (Nelson, 1967) or the Parisi and Wu method (Damgaard, 1987; Parisi and Wu, 1982), the HK approach seems to be remarkably concise. Moreover, as shown in Haba and Kleinert (2002), the HK method can be applied to arbitrary one-dimensional mechanical systems subject to a time-independent potential.

In view of the elegance and conciseness of the HK method, it is valuable to extend it to more general dynamical systems. Several avenues are open in this regard. For instance, one can address the generalization of HK to higher dimensionality, to composite systems, to systems exposed to external electromagnetic fields and so on. Here we focus on the stochastic quantization of explicitly time-dependent systems. In fact, as long as we know stochastic quantization methods have not paid much attention to non-autonomous systems, in spite of the importance of these for applications. We consider the use of HK for a particular non-autonomous system, namely, the time-dependent harmonic oscillator (TDHO), characterized by a frequency function variable along time (Lewis, 1967). We refrain from listing the large list of applications of the TDHO, including such several fields as quantum optics, cosmology, non-linear elasticity and hydrodynamics (see Espinoza, 2000, for a review). Here we limit ourselves to apply HK to this class of systems, showing that some non-autonomous systems are also amenable to stochastic quantization via the HK method. We will see that the time-dependence of the frequency can be eliminated through a rescaling transformation. Other non-autonomous systems may perhaps also be treated through coordinate transformations, but this possibility is unproven presently. The work presents the first non-autonomous class of mechanical systems amenable to stochastic quantization by the HK approach, namely, the TDHO system.

The paper is organized as follows. In Section II, we apply the HK method to the TDHO equation. In this section, we both present the HK method with some more details than in the original reference (Haba and Kleinert, 2002) and apply it, obtaining a stochastic differential equation simulating the TDHO. In Section III, we obtain the formal solution for this stochastic differential equation. This formal solution is used to describe the time-evolution of the mother field, one of the fundamental elements in the HK method. We then derive the Schrödinger equation associated to the TDHO. Section IV is devoted to the conclusions.

2. STOCHASTIC DIFFERENTIAL EQUATION

Our purpose is to apply the HK method to obtain the quantization of the TDHO equation,

$$\ddot{x} + \omega^2(t)\,x = 0\,, \tag{1}$$

where $\omega(t)$ is the time-dependent frequency. The HK method consists of three steps:(a) postulate of a classical, stochastic equation, which reduces to the Newton equation of the system in the limit of zero noise; (b) introduction of an auxiliary field (the mother field) defined in terms of the solutions of the original, stochastic equation; (c) averaging over stochastic processes, yielding a deterministic equation equivalent to the Schrödinger equation. For the TDHO, we choose the stochastic equation

$$\dot{\mathbf{r}} = \frac{\mathbf{\Omega} \times \mathbf{r}}{\rho^2} + \frac{\dot{\rho}}{\rho}\,\mathbf{r} + f(t)\,\mathbf{n}\,, \tag{2}$$

where $\mathbf{r} = (x, y, 0)$ and $\mathbf{\Omega} = (0, 0, \Omega)$ for constant Ω, and where $\mathbf{n} = (1, 1, 0)/\sqrt{2}$ is a unitary vector in the diagonal direction in the plane. The function ρ is any particular solution for the Pinney's (1950) equation,

$$\ddot{\rho} + \omega^2(t)\rho = \frac{\Omega^2}{\rho^3}\,, \tag{3}$$

where $\omega(t)$ is the time-dependent frequency characterizing the TDHO. Pinney equations also appear in other contexts like Bose–Einstein condensate dynamics (Haas, 2002) or non-relativistic charged particle motion (Haas and Goedert, 1999). It is convenient to take $\Omega \neq 0$ so that ρ never vanish. Finally, $f(t)$ is a stochastic variable with zero mean over statistical processes,

$$\langle f(t) \rangle = 0\,, \tag{4}$$

and with correlation function

$$\langle f(t) f(t') \rangle = \hbar\,\delta(t - t')\,, \tag{5}$$

so there is a white noise stochastic process.

The choice for the classical dynamical equation is justified because, treating formally f as an ordinary function, the second-order equation obtained from (2) is

$$\ddot{\mathbf{r}} + \omega^2(t)\,\mathbf{r} = \left(\dot{f} + \frac{\dot{\rho}}{\rho}\,f\right)\mathbf{n} + \frac{f}{\rho^2}\,\mathbf{\Omega} \times \mathbf{n}\,, \tag{6}$$

which reduces to the planar isotropic TDHO when $f = 0$. Latter on, as we shall see, the HK method proposes a dimensional reduction from 2D to 1D, so that we will recover the one-dimensional TDHO.

To proceed, let us introduce the well-known rescaling (Munier *et al.*, 1981),

$$X = \frac{x}{\rho}\,, \quad Y = \frac{y}{\rho}\,, \quad T = \int_0^t \frac{dt'}{\rho^2}(t')\,. \tag{7}$$

With the new variables, we obtain from (2) the rescaled stochastic equation

$$\frac{d\mathbf{R}}{dT} = \mathbf{\Omega} \times \mathbf{R} + F(T)\,\mathbf{n}\,, \tag{8}$$

where $\mathbf{R} = (X, Y, 0)$ and $F(T)$ is defined by

$$F(T) = \rho(t)\, f(t)\,. \tag{9}$$

Since f is a stochastic function, so is F. The statistical properties of F follows from (4–5). The new stochastic function has zero mean,

$$\langle F(T)\rangle = 0\,, \tag{10}$$

and has correlation function

$$\langle F(T)F(T')\rangle = \hbar\,\delta(T - T'), \tag{11}$$

defining a white noise.

Equation (10) is an immediate consequence of the zero mean of f. On the other hand (11), is demonstrated in the following way. By the definition of F, we have

$$\langle F(T)F(T')\rangle = \rho(t)\rho(t')\langle f(t)f(t')\rangle, \tag{12}$$

where the transformed times are

$$T = \int_0^t \frac{dt''}{\rho^2(t'')}, \quad T' = \int_0^{t'} \frac{dt''}{\rho^2(t'')}\,. \tag{13}$$

From (12) and the correlation function in terms of the original variables, we obtain

$$\langle F(T)F(T')\rangle = \hbar\rho^2(t)\,\delta(t - t')\,. \tag{14}$$

This gives an expression for the correlation function of F in terms of the original time variable. Our objective is to show that the right-hand side of (11) coincides with this.

It turns out that

$$\delta(T - T') = \delta\left(\int_{t'}^{t} \frac{dt''}{\rho^2(t'')}\right)\,, \tag{15}$$

using the definition of rescaled time. The right-hand side of the last equation can be handled with the following property of the delta function,

$$\delta(\varphi(t)) = \sum_i \frac{\delta(t - t_i)}{|\varphi'(t_i)|}\,, \tag{16}$$

for an arbitrary function φ and for $\varphi(t_i) = 0$, $\varphi'(t_i) \neq 0$. The sum (16) is over the zeros of the function φ. Applying (16) to (15), we get

$$\delta(T - T') = \rho^2(t)\,\delta(t - t')\,, \tag{17}$$

thus showing the equivalence between (11) and (14) as we desired.

In the new variables, Equation (8) is the same as that used by HK in the case of the time-independent harmonic oscillator (see Eq. (3) of Haba and Kleinert, 2002). Hence, our remaining task is to repeat the procedure by HK and map our conclusions to the original, non-rescaled variables. We also offer some extra details on the necessary calculations, not present in the original work of HK. In particular, we consider in more detail the expansion procedure of HK.

3. MOTHER FIELD AND STOCHASTIC QUANTIZATION

We can trivially obtain the formal solution for Equation (8) as follows. Thinking the stochastic function F as an ordinary function, we easily obtain the linear invariants

$$X_0 = X\cos\Omega T + Y\sin\Omega T - \frac{1}{\sqrt{2}}\int_0^T dT'\, F(T')(\cos\Omega T' + \sin\Omega T'), \tag{18}$$

$$Y_0 = -X\sin\Omega T + Y\cos\Omega T - \frac{1}{\sqrt{2}}\int_0^T dT'\, F(T')(\cos\Omega T' - \sin\Omega T') \tag{19}$$

for the dynamical equations. In other words, both X_0 and Y_0 are constant along trajectories and can be interpreted as the initial conditions for X and Y respectively. Solving (18–19) for X and Y we obtain

$$X = X_0\cos\Omega T - Y_0\sin\Omega T + \frac{\cos\Omega T}{\sqrt{2}}\int_0^T dT' F(T')(\cos\Omega T' + \sin\Omega T') - \frac{\sin\Omega T}{\sqrt{2}}\int_0^T dT' F(T')(\cos\Omega T' - \sin\Omega T'), \tag{20}$$

$$Y = Y_0\cos\Omega T + X_0\sin\Omega T + \frac{\cos\Omega T}{\sqrt{2}}\int_0^T dT' F(T')(\cos\Omega T' - \sin\Omega T') + \frac{\sin\Omega T}{\sqrt{2}}\int_0^T dT' F(T')(\cos\Omega T' + \sin\Omega T'). \tag{21}$$

We can use the exact solution to expand X and Y around $T = 0$. Integrating (11) we get the estimate $F^2(\Delta T) \sim \hbar/\Delta T$ for small ΔT. Therefore, $\int_0^{\Delta T} F(T)dT \sim \sqrt{\hbar\Delta T}$. Using this and expanding the exact solution, the result is

$$X = X_0 - \Omega Y_0\Delta T + \frac{1}{\sqrt{2}}\int_0^{\Delta T} dT\, F(T) + \mathrm{O}((\Delta T)^{3/2}), \tag{22}$$

$$Y = Y_0 + \Omega X_0 \Delta T + \frac{1}{\sqrt{2}} \int_0^{\Delta T} dT\, F(T) + \mathrm{O}((\Delta T)^{3/2}). \tag{23}$$

Hence we deal with an expansion in powers of $\varepsilon = (\Delta T)^{1/2}$. More precisely,

$$X = X_0 + \varepsilon \alpha_1 + \varepsilon^2 \alpha_2 + \mathrm{O}(\varepsilon^3), \tag{24}$$

$$Y = Y_0 + \varepsilon \beta_1 + \varepsilon^2 \beta_2 + \mathrm{O}(\varepsilon^3), \tag{25}$$

where

$$\varepsilon \alpha_1 = \varepsilon \beta_1 = \frac{1}{\sqrt{2}} \int_0^{\Delta T} dT\, F(T)\,, \tag{26}$$

$$\varepsilon^2 \alpha_2 = -\Omega Y_0\, \Delta T\,, \quad \varepsilon^2 \beta_2 = \Omega X_0\, \Delta T\,. \tag{27}$$

Following HK, the next step is the introduction of an auxiliary field, depending on the solution for the stochastic equation (8). Let us define a time-independent mother field $\mathbf{U}(\mathbf{R}) = (U_1(\mathbf{R}), U_2(\mathbf{R}))$ supposed to satisfy the Cauchy–Riemann conditions

$$\frac{\partial U_1}{\partial X} = \frac{\partial U_2}{\partial Y}\,, \quad \frac{\partial U_1}{\partial Y} = -\frac{\partial U_2}{\partial X}\,. \tag{28}$$

At time T, we can construct a time-dependent field $\mathbf{U}_T(\mathbf{R}, T)$ which at $T = 0$ equals $\mathbf{U}(\mathbf{R})$ and which is given in terms of the flow of the stochastic differential equation (8),

$$\mathbf{U}_T(\mathbf{R}, T) = \mathbf{U}(\mathbf{R}(T))\,, \tag{29}$$

where $\mathbf{R}(T)$ evolves in time according to (8). Using (24–25), we can expand the time-dependent field around $T = 0$ to obtain

$$\mathbf{U}_T(\mathbf{R}, \Delta T) = \mathbf{U}(\mathbf{R}) + \Delta T\, (\boldsymbol{\Omega} \times\, \mathbf{R} \cdot \bar{\nabla})\mathbf{U}(\mathbf{R}) + \int_0^{\Delta T} dT\; F(T)(\mathbf{n} \cdot \bar{\nabla})\mathbf{U}(\mathbf{R}) + \frac{1}{2} \int_0^{\Delta T} dT \int_0^{\Delta T} dT' F(T) F(T') (\mathbf{n} \cdot \bar{\nabla})^2\, \mathbf{U}(\mathbf{R}) + \mathrm{O}((\Delta T)^{3/2})\,. \tag{30}$$

where $\bar{\nabla}$ is the gradient operator in $\mathbf{R}$ coordinates.

By construction, $\mathbf{U}_T(\mathbf{R}, T)$ is a stochastic function. However, an ordinary function $\bar{\mathbf{U}}(\mathbf{R}, T)$ can be build after performing the last step in the HK method, i.e., averaging over stochastic processes,

$$\bar{\mathbf{U}}(\mathbf{R}, T) = \langle \mathbf{U}_T(\mathbf{R}, T) \rangle\,. \tag{31}$$

Averaging (30) and taking into account the statistics of F, we obtain

$$\frac{\partial \bar{\mathbf{U}}}{\partial T} = \hat{H}\,\bar{\mathbf{U}}\,, \tag{32}$$

with the time evolution operator

$$\hat{H} = ((\boldsymbol{\Omega} \times \mathbf{R}) \cdot \bar{\nabla}) + \frac{\hbar}{2}(\mathbf{n} \cdot \bar{\nabla})^2. \tag{33}$$

As shown by HK, the averaged mother field is harmonic for all times.

Of course most of the steps followed until now are precisely the same as those presented at (Haba and Kleinert, 2002), in connection with the time-independent harmonic oscillator. The non-trivial content of this work is the use of a coordinate transformation for the treatment of the TDHO by stochastic quantization. This should be not underestimated, since no other time-dependent system was ever shown to be amenable to HK quantization. Moreover, we think that showing the details of the calculations is necessary in order to provide a precise translation from the time-independent to the time-dependent scenarios.

Equation (32) is equivalent to a Schrödinger equation after restricting to the $Y = 0$ line. To show this, notice that the averaging process preserves the harmonic property (28). Using the Cauchy–Riemann properties, we can rewrite (32) as

$$\frac{\partial \bar{U}_1}{\partial T} = -\,\Omega\left(X\frac{\partial \bar{U}_2}{\partial X} + Y\frac{\partial \bar{U}_1}{\partial X}\right) - \frac{\hbar}{2}\frac{\partial^2 \bar{U}_2}{\partial X^2}\,, \tag{34}$$

$$\frac{\partial \bar{U}_2}{\partial T} = \Omega\left(X\frac{\partial \bar{U}_1}{\partial X} - Y\frac{\partial \bar{U}_2}{\partial X}\right) + \frac{\hbar}{2}\frac{\partial^2 \bar{U}_1}{\partial X^2}\,. \tag{35}$$

Restricting to $Y = 0$ and defining the complex field

$$\bar{\psi}(X,T) = \exp\!\left(-\frac{\omega X^2}{2\hbar} - \frac{i\omega T}{2}\right)(\bar{U}_1(X,0,T) + i\bar{U}_2(X,0,T))\,, \tag{36}$$

we obtain the Schrödinger equation for a one-dimensional time-independent harmonic oscillator,

$$i\hbar\frac{\partial \bar{\psi}}{\partial T} = \left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial X^2} + \frac{\Omega^2 X^2}{2}\right)\bar{\psi}\,, \tag{37}$$

with frequency Ω.

To obtain the quantization of the TDHO, consider the transformation

$$\psi = \rho^{-1/2}\exp\!\left(\frac{i\dot{\rho}\,x^2}{2\hbar\rho}\right)\bar{\psi}\,, \tag{38}$$

where ρ is a solution for the Pinney equation (3). Using the Pinney equation and (37) and inverting the rescaling (7), we get

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2} \frac{\partial^2}{\partial x^2} + \frac{\omega^2(t) x^2}{2} \right) \psi \,. \tag{39}$$

The solution of the quantum one-dimensional TDHO (39) can be obtained, for instance, obtaining a particular solution for the Pinney equation and applying the Lewis–Riesenfeld method (Lewis and Riesenfeld, 1969).

4. CONCLUSION

We have obtained the quantization of the TDHO using the HK method. Our approach relies on a rescaling transformation, which removes the explicit time-dependence from the classical stochastic Newtonian equation. We may observe that, since the quantization procedure depends on the function $\rho(t)$, which is any arbitrary solution for the Pinney equation, we have in fact an infinite family of quantizations. For each initial condition of the Pinney equation we have a different stochastic quantization. We have presented a dictionary relating the time-dependent and the time-independent scenarios, namely, the coordinate transform, the wavefunction transform and the correlation functions for both the cases.

Since, essentially, our procedure was based on a rescaling transformation, there remains the question of what other classes of explicitly time-dependent mechanical systems are amenable to the HK method. We have not shown, in particular, that arbitrary time-dependent potential functions can be treated by HK. In our opinion this is an important issue concerning the HK approach. The fact that the TDHO can be handled by a coordinate transformation is a non-trivial fact that singles out the special role of this system. No nonlinear time-dependent mechanical system was, until now, shown to be tractable by HK. In addition, we have not touched upon the measurement theory associated to HK or the physical origin of the noise (as in most stochastic quantization methods). Finally, in the case of HK there are other possible and necessary extensions, like consideration of higher dimensional cases, many particle systems, and the inclusion of general electromagnetic fields.

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